On the asymptotic behavior of solutions of nonlinear Volterra equations and its application to nonlinear heat flow with memory

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1. Introduction.

We shall consider the problem of nonlinear heat flow in materials with memory:

$$
(\mathrm{M})\begin{cases}
\dfrac{\partial}{\partial t}\left(u(t,x)+\displaystyle\int_{-\infty}^{t} k(t-s)u(s,x)\,ds\right) = \sigma(u_x(t,x))_x + h(t,x), \\
\qquad\qquad\qquad\qquad\qquad\qquad t\in R^+,\ x\in(0,1), \\
u_x(t,0)\in \beta_0(u(t,0)),\ -u_x(t,1)\in\beta_1(u(t,1)),\ t\in R, \\
u(t,x)=u_0(x),\quad t\in(-\infty,0),\ x\in(0,1).
\end{cases}
$$

Our main objective is to show the existence of a " generalized " solution of (M) and its asymptotic behavior. We interprete (M) as an abstract nonlinear Volterra equation in $L^p(0,1)$ of the form:

$$
(\mathrm{E})\begin{cases}
\dfrac{d}{dt}u(t)+Au(t)+G(u)(t)\ni h(t)+k(t)u_0, \quad t\in R^+, \\
u(0)=u_0,
\end{cases}
$$

which can be rewritten as

$$
u(t)+\int_0^t b(t-s)Au(s)\,ds\ni g(t), \quad t\in R^+,
$$

with appropriate functions b and g (cf. [5]).

In §2 we prepare an abstract theory dealing with (E), after that in §3 we state the main result of this paper (Theorem 3.3) and some remarks, and the proofs are contained in §4.

2. Abstract results.

In this section, let $(X, \|\cdot\|)$ be a real Banach space and A be an operator in X, and we consider the following evolution equation:

$$(E)\quad \begin{cases} \frac{d}{dt} u(t)+Au(t)+G(u)(t) \ni h(t)+k(t)u_0, & t \in R^+, \\ u(0)=u_0, \end{cases}$$

where $G(u)(t)=k(0)u(t)+\int_0^t u(t-s)\,dk(s)$, $k \in BV_{loc}(R^+)$ and $h(t) \in L^1_{loc}(R^+;X)$. By a strong solution of (E), we mean a function in $W^{1,1}_{loc}(R^+;X) \cap C(R^+;\overline{D(A)})$ which satisfies (E) for a.e. $t \in R^+$. A function $u \in C(R^+;\overline{D(A)})$ is called simply a solution if it is an "integral solution" of (E) considering $h(t)+k(t)u-G(u)(t)$ as an inhomogeneous term. For the existence of a solution of (E), we recall the following

<u>THEOREM 2.1 ([5]).</u> Assume that A is m-accretive, $u_0 \in \overline{D(A)}$, and $h \in L^1_{loc}(0,\infty;X)$. Then there exists a unique solution $u(t)$ of (E). If X is reflexive, $h \in BV_{loc}(R^+;X)$ and $u_0 \in D(A)$, then u is a strong solution.

Now, we consider the asymptotic behavior. For the sake of simple and unified treatment, let X be uniformly convex and smooth. Then we can define the continuous nearest point mapping P onto $A^{-1}0$, provided $A^{-1}0 \neq \Phi$. Denote by J the single-valued duality mapping.

Definition ([11]). A is said to satisfy the convergence condition if $(x_n, y_n) \in A$, $\|x_n\| \leq M$, $\|y_n\| \leq M$ and $\lim_{n\to\infty} \langle y_n, J(x_n - Px_n)\rangle = 0$ imply $\underline{\lim}_{n\to\infty} \|x_n - Px_n\| = 0$.

Concerning the asymptotic behavior, we have the

THEOREM 2.2. Let $k \in L^1(R^+)$, nonnegative, nonincreasing, and bounded. Let $h \in L^1(R^+;X)$ and $u_0 \in \overline{D(A)}$. Assume that A is m-accretive, $A^{-1}0 \neq \Phi$, and A satisfies the convergence condition. Then

(2.1) $\lim_{t\to\infty} \|Pu(t+h) - Pu(t)\| = 0$ for each fixed $h > 0$

implies that the solution u(t) of (E) converges strongly to an element of $A^{-1}0$ as $t \to \infty$.

It can be shown that if $(I+A)^{-1}$ is compact, then $\lim_{t\to\infty} Pu(t)$ exists, and hence (2.1) is valid. Therefore we obtain the

COROLLARY 2.3. Let k, h, u_0, and A be as above. If $(I+A)^{-1}$ is compact, then the solution u(t) of (E) converges strongly to an element of $A^{-1}0$ as $t \to \infty$.

Remark. Indeed, we can assume the weaker condition than the compactness of $(I+A)^{-1}$, the bounded compactness of $A^{-1}0$ as (13), but the above setting is available from the viewpoint of proposition below, which states the sufficient condition for A to satisfy the convergence condition.

PROPOSITION 2.4 ([11,12]). Let X be uniformly convex and smooth. Let A be m-accretive with $A^{-1}0 \neq \Phi$. If $\langle y, J(x-Px)\rangle > 0$ for every $(x,y) \in A$ with $x \notin A^{-1}0$, and the resolvent $(I+A)^{-1}$ is compact, then A satisfies the convergence condition.

Proof of Theorem 2.2. Note that $Pu(\cdot) \in L^{\infty}(R^{+};X)$ by the boundedness of $u(t)$ (since $A^{-1}0 \neq \Phi$). Using [10, (3.1)] or [7, Lemma 2.10(a)], it may be assumed that $u_0 \in D(A)$, $h \in C^1(R^{+};X) \cap W^{1,1}(R^{+};X)$ and $u(t)$ is a strong solution of (E). Then [7] shows that

$$\lim_{t\to\infty} \|u(t)-Pu(t)\| = 0 \tag{2.2}$$

by using the convergence condition. $\|u(t+h)-u(t)\| \leq \|u(t+h)-Pu(t)\| + \|Pu(t)-u(t)\|$ and

$$\begin{aligned} \|u(t+h)-Pu(t)\| &\leq \|u(t)-Pu(t)\| + \int_0^t k(t-\tau)\|u(\tau)-Pu(t)\|\,d\tau \\ &\quad + \int_t^{t+h} \|h(\tau)+k(\tau)g(0)-k(\tau)Pu(t)\|\,d\tau \\ &\leq \|u(t)-Pu(t)\| + \int_0^t k(t-\tau)\|u(\tau)-Pu(t)\|\,d\tau \\ &\quad + \int_t^{\infty}\|h(\tau)\|\,d\tau + M\int_t^{\infty} k(\tau)\,d\tau, \end{aligned}$$

by [10,Lemma 3.1]. Thus if we show that

$$(2.3)\quad \lim_{t\to\infty}\int_0^t k(t-\tau)\|u(\tau)-Pu(t)\|\,d\tau = 0,$$

then the proof will be complete. Now, fix $T>0$ and let $t>T$.

$$\begin{aligned}
&\int_0^t k(t-\tau)\|u(\tau)-Pu(t)\|\,d\tau \\
&= \int_0^{t-T} k(t-\tau)\|u(\tau)-Pu(t)\|\,d\tau + \int_{t-T}^t k(t-\tau)\|u(\tau)-Pu(t)\|\,d\tau \\
&\leq M\int_T^\infty k(\eta)\,d\eta + \sup_{t-T\leq\xi<\infty}\|u(\xi)-Pu(\xi)\|\int_0^\infty k(\eta)\,d\eta \\
&\quad + \int_0^t k(\eta)\|Pu(t-\eta)-Pu(t)\|\,d\eta \\
&\leq \tilde{M}\Big(\int_T^\infty k(\eta)\,d\eta + \sup_{t-T\leq\xi<\infty}\|u(\xi)-Pu(\xi)\| + \int_0^T\|Pu(t-\eta)-Pu(t)\|\,d\eta\Big).
\end{aligned}$$

By (2.2), (2.1) and the Lebesgue convergence theorem, we have $\overline{\lim_{t\to\infty}}\int_0^t k(t-\tau)\|u(\tau)-Pu(t)\|\,d\tau \leq \tilde{M}\int_T^\infty k(\eta)\,d\eta$. Since T is arbitrary, (2.3) is proved. □

<u>Proof of Cororally 2.3.</u> Noting that $\{Pu(t): t\geq 0\}$ is bounded and $Pu(t)=(I+A)^{-1}Pu(t)$, since $(I+A)^{-1}$ is compact, $\{Pu(t): t\geq 0\}$ is precompact. Thus there exist $t_n\to\infty$ such that $\lim_{n\to\infty}Pu(t_n)=\alpha$ for some $\alpha\in A^{-1}0$. Let $r(y)=\overline{\lim_{t\to\infty}}\|u(t)-y\|$ for any $y\in A^{-1}0$ (indeed, limit exists by [10, Lemma 3.2]). In general if X is uniformly convex, then there exists uniquely $z\in A^{-1}0$ such that $r(z)=\inf\{r(y): y\in A^{-1}0\}$. (See e.g. [6, Chap.1, Th.4.1].) It then follows that

$$r(\alpha)=\overline{\lim_{n\to\infty}}\|u(t_n)-\alpha\| \leq \overline{\lim_{n\to\infty}}(\|u(t_n)-z\| + \|Pu(t_n)-\alpha\|) = r(z).$$

Thus $\alpha = z$ and $\lim_{t\to\infty}Pu(t) = z$. This completes the proof. □

3. Nonlinear heat flow with memory.

In this section, we consider the following problem of nonlinear heat flow in materials with memory:

$$
(\mathrm{M})\quad \begin{cases} \dfrac{\partial}{\partial t}\left(u(t,x)+\displaystyle\int_{-\infty}^{t} k(t-s)u(s,x)\,ds\right) = \sigma(u_x(t,x))_x + h(t,x), \\ \hfill t\in R^+,\ x\in(0,1), \\ u_x(t,0)\in \beta_0(u(t,0)),\ -u_x(t,1)\in\beta_1(u(t,1)),\ t\in R, \\ u(t,x)=u_0(x),\quad t\in(-\infty,0),\ x\in(0,1). \end{cases}
$$

where $k\in L^1(R^+)\cap BV_{loc}(R^+)$ and σ satisfies

(σ) $\sigma\in C^1(R)$, $\sigma(0)=0$, $\sigma(R)=R$, $\sigma'(r)>0$ $(r\in R)$,

and β_i $(i=0,1)$ are maximal monotone graphs in $R\times R$ satisfying $0\in\beta_i(0)$.

Examples:

1. $\beta_i=0$ $(i=0,1)$ $\Rightarrow$ Neumann condition.
2. $\beta_i(x)=\begin{cases} R & \text{if } x=0 \\ \Phi & \text{if } x\neq 0\end{cases}$ $(i=0,1)$ $\Rightarrow$ Dirichlet condition.
3. $\beta_0=0$ and $\beta_1(x)=\begin{cases} R & \text{if } x=0 \\ \Phi & \text{if } x\neq 0\end{cases}$ $\Rightarrow$ $u'(0)=0$ and $u(1)=0$.

Let $1<p<\infty$. In order to interpret (M) as an abstract equation (E), define L by

$$D(L)=\{u\in C^2[0,1]:\ u'(0)\in\beta_0(u(0)),\ -u'(1)\in\beta_1(u(1))\}$$

$$Lu=-\sigma(u')'\qquad \text{for } u\in D(L),$$

and then, considering $L: D(L)\subset L^p(0,1)\to L^p(0,1)$, let

$$A = L^p\text{-closure of } L.$$

Since $u(t,x)=u_0(x)$ for $t \in (-\infty, 0)$,

$$\frac{\partial}{\partial t}\left(\int_{-\infty}^{t} k(t-s)u(s,x)\,ds\right) = \frac{\partial}{\partial t}\left(\int_{0}^{t} k(t-s)u(s,x)\,ds + \int_{t}^{\infty} k(s)\,ds\; u_0(x)\right)$$

$$= k(0)u(t,x)+\int_{0}^{t} u(t-s,x)\,dk(s) - k(t)u_0(x).$$

Therefore, we can see (M) as (E).

Our main aim is to show the following two propositions:

PROPOSITION 3.1. Suppose that

(3.1) $\quad \int_0^\infty r \min\{\sigma'(s) : |s| \leqq r\}\,dr = \infty$, and

(3.2) $\quad \sup\{|y| : y \in R(\beta_i)\} < \infty$ for $i=0$ or 1

Then A is an m-accretive operator in $L^p(0,1)$, the resolvent $(I+A)^{-1}$ is a compact operator, and A satisfies the convergence condition defined in §2.

PROPOSITION 3.2. Suppose that

(3.3) $\quad \exists \delta > 0: \sigma'(r) \geq \delta$ (Note that (3.3) $\Rightarrow$ (3.1).)

Then the same conclusions as above hold.

Once the above propositions are obtained, we can apply the abstract results Theorem 2.1 and Corollary 2.3 to obtain the existence of a unique generalized solution of (M) and its asymptotic behavior, where generalized solution of (M) means the (integral) solution of (E) when we interpret (M) as (E), with A defined above.

<u>THEOREM 3.3.</u> Let $k\in L^1(R^+)$, nonnegative, nonincreasing and bounded. Let $h\in L^1(R^+;L^p(0,1))$ and $u_0\in L^p(0,1)$, $1<p<\infty$. Assuming either (3.1) and (3.2), or (3.3), then the unique generalized solution $u(t,x)$ of (M) exists and it converges strongly in $L^p(0,1)$ to some constant ξ_∞ satisfying $0\in \beta_i(\xi_\infty)$ $(i=0,1)$.

<u>Remarks</u> 1. In the case of Neumann boundary condition, (3.1) is unnecessary as will be shown in [9], and it is easy to see that

$$\xi_\infty = \int_0^1 u_0(x)\,dx + \left(1+\int_0^\infty k(s)\,ds\right)^{-1}\int_0^\infty\int_0^1 h(t,x)\,dx\,dt \quad \text{(cf. [2])}.$$

2. In the case of Dirichlet boundary condition, we don't know (3.3) can be removable. (For A to be m-accretive, it sufficies to assume only (3.1) as shown in Lemma 4.2.) If the Dirichlet boundary condition and (3.3) are assumed, A becomes strongly accretive by the Poincaré inequality and if k is as above, we can obtain the estimate of decay ([4], [8]):

$$(3.4) \qquad \|u(t)\|_p \le \left(\int_t^\infty r(\tau)\,d\tau\right)\|u_0\|_p + \omega^{-1}\int_0^t r(t-\tau)[u(\tau),h(\tau)]_+\,d\tau,$$

where $\omega>0$ is a constant for which $A-\omega I$ is accretive, and r is defined by $r+\omega b*r=\omega b$, $b+k*b=1$, and $[x,y]_+=\lim_{\lambda\downarrow 0}(\|x+\lambda y\|-\|x\|)/\lambda$. It is known [4] that $r\ge 0$ and $r\in L^1(0,\infty)$. Observe that if $k\equiv 0$, then $\int_t^\infty r(\tau)\,d\tau = \frac{C}{\omega}e^{-\omega t}$. Thus (3.4) corresponds to an exponential decay.

4. Proofs of Proposition 3.1 and 3.2.

The proofs are established by a series of lemmas below.

Lemma 4.1. A is accretive in $L^p(0,1)$.

Proof. It sufficies to prove the accretiveness of L in $L^p(0,1)$.

$$\|u-v\|^{p-1}[u-v,Lu-Lv]_+ = -\int_0^1 |u-v|^{p-1}\mathrm{sgn}(u-v)(\sigma(u')'-\sigma(v')')$$
$$= (-|u-v|^{p-1}\mathrm{sgn}(u-v)(\sigma(u')-\sigma(v')))_0^1$$
$$+\int_0^1 (p-1)|u-v|^{p-2}(\mathrm{sgn}(u-v))^2(u'-v')(\sigma(u')-\sigma(v'))$$
$$= -|u(1)-v(1)|^{p-2}(u(1)-v(1))(\sigma(u'(1))-\sigma(v'(1)))$$
$$+|u(0)-v(0)|^{p-2}(u(0)-v(0))(\sigma(u'(0))-\sigma(v'(0)))$$
$$+\int_0^1 (p-1)|u-v|^{p-2}(u'-v')(\sigma(u')-\sigma(v'))$$

Since $-u'(1)\in\beta_1(u(1))$, $-v'(1)\in\beta_1(v(1))$, β_1 is monotone and σ is increasing, $\mathrm{sgn}(\sigma(u'(1))-\sigma(v'(1)))=\mathrm{sgn}(u'(1)-v'(1))=-\mathrm{sgn}(u(1)-v(1))$. Thus $(u(1)-v(1))(\sigma(u'(1))-\sigma(v'(1)))\leq 0$. Similarly, $(u(0)-v(0))(\sigma(u'(0))-\sigma(v'(0)))\geq 0$. Also, since σ is increasing, $(u'-v')(\sigma(u')-\sigma(v'))\geq 0$. Hence $[u-v,Lu-Lv]_+\geq 0$. □

Lemma 4.2. Assume (3.1). Then A is m-accretive in $L^p(0,1)$.

Proof. (14) shows that L is m-accretive in $C[0,1]$. Hence

$$C[0,1]=R(I+L)\subset R(I+A)\subset L^p(0,1).$$

By Lemma 4.1, A is accretive in $L^p(0,1)$, and by the definition it is closed in $L^p(0,1)$. Hence $R(I+A)$ is a closed subset of $L^p(0,1)$, and so $L^p(0,1)=R(I+A)$. □

<u>Lemma 4.3.</u> Assume (3.2). Then

$D(A) \subset \{u \in W^{2,p}(0,1);\ u'(0) \in \beta_0(u(0)),\ -u'(1) \in \beta_1(u(1))\}$, and

$Au = -\sigma(u')'$ for $u \in D(A)$.

Proof. Suppose (3.2) holds for i=0. Let $(u,v) \in A$. Then there exist $u_n \in D(L)$ such that $u_n \to u$ in $L^p(0,1)$ and $-\sigma(u_n')' \to v$ in $L^p(0,1)$. Since $u_n'(0) \in \beta_0(u_n(0))$, one can extract a subsequence $\{n_k\} \subset \{n\}$ such that $u_{n_k}'(0) \to {}^{\exists}w \in R$. For simplicity, denote n_k by n again. Noting that

(4.1) $$\sigma(u_n'(x)) - \sigma(u_n'(0)) = \int_0^x \sigma(u_n'(\tau))' d\tau,$$

we have

(4.2) $$|u_n'(x)| \leq C$$

Furthermore, (4.1) and the continuity of σ^{-1} imply that $u_n'(x) \to \sigma^{-1}(\sigma(w) - V(x))$, $x \in [0,1]$, where $V(x) = \int_0^x v(\tau) d\tau$. Therefore, $u_n' \to \sigma^{-1}(\sigma(w) - V(\cdot))$ in $L^p(0,1)$. It then follows that $u \in W^{1,p}(0,1)$ and $u' = \sigma^{-1}(\sigma(w) - V)$. Since $V \in W^{1,p}(0,1)$, we have $u \in W^{2,p}(0,1)$ and $\sigma(u')' = -v$. Thus A is single-valued and $Au = -\sigma(u')'$ for $u \in D(A)$.

Finally, we check the boundary condition. Since $\sigma(u_n')' \to -Au = \sigma(u')'$ in L^p, there exist $n_j \to \infty$ such that $\sigma(u_{n_j}'(x))' \to \sigma(u'(x))'$ a.e. $x \in (0,1)$ and $|\sigma(u_{n_j}'(x))'| \leq h(x), {}^{\forall}j$, a.e. $x \in (0,1)$ for some $h \in L^p(0,1)$. (see e.g. [3, Theorem IV9].) Then observing $\sigma^{-1} \in C^1(R)$ and $u_n''(x) = (\sigma^{-1})'(\sigma(u_n'(0)) + \int_0^x \sigma(u_n'(\tau))' d\tau)\ \sigma(u_n'(x))'$, we have $u_{n_j}''(x) \to (\sigma^{-1})'(\sigma(w) + \int_0^x \sigma(u'(\tau))' d\tau) \sigma(u'(x))' = u''(x)$ a.e. x, and

$|u''_{n_j}(x)| \leq Mh(x)$. (Here note that $|\int_0^x \sigma(u')'d\tau| \leq \int_0^1 |\sigma(u'_n)'|\,d\tau \to \int_0^1 |\sigma(u')'|\,d\tau$, (4.2), and $\sigma^{-1} \in C^1(R)$ imply that $|(\sigma^{-1})'(\sigma(u'_{n_j}(0)) + \int_0^x \sigma(u'_{n_j})'d\tau| \leq M$.) Thus by the Lebesgue convergence theorem, $u''_{n_j} \to u''$ in L^p, so that $u_n \to u$ in $W^{2,p}(0,1) \subsetneq C^1[0,1]$ and then $u'_n(0) \to u'(0)$, $u_n(0) \to u(0)$, $u'_n(1) \to u'(1)$, and $u_n(1) \to u(1)$. By the closedness of β_i $(i=0,1)$, we conclude that $u'(0) \in \beta_0(u(0))$ and $-u'(1) \in \beta_1(u(1))$. If (3.2) holds for $i=1$, then instead of (4.1), we use $\sigma(u'_n(1)) - \sigma(u'_n(x)) = \int_x^1 \sigma(u'(\tau))'d\tau$. □

Lemma 4.4. Assume (3.1) and (3.2). Then

$D(A) = \{u \in W^{2,p}(0,1);\ u'(0) \in \beta_0(u(0)),\ -u'(1) \in \beta_1(u(1))\}$, and

$Au = -\sigma(u')'$ for $u \in D(A)$.

Proof. Define $B: L^p \to L^p$ by

$D(B) = \{u \in W^{2,p}(0,1);\ u'(0) \in \beta_0(u(0)),\ -u'(1) \in \beta_1(u(1))\}$, and

$Bu = -\sigma(u')'$ for $u \in D(B)$.

As Lemma 4.1, B is accretive in $L^p(0,1)$ and by Lemma 4.3, $A \subset B$. Since A is m-accretive in $L^p(0,1)$ by Lemma 4.2, the maximality of A implies $A = B$. □

Lemma 4.5. Assume (3.3). Then

$D(A) = \{u \in W^{2,p}(0,1);\ u'(0) \in \beta_0(u(0)),\ -u'(1) \in \beta_1(u(1))\}$, and

$Au = -\sigma(u')'$ for $u \in D(A)$.

Proof. Let $(u,v) \in A$. Then there exist $u_n \in D(L)$ such that $u_n \to u$ in L^p and $-\sigma(u_n')' \to v$ in L^p. Hence $\|\sigma'(u_n')' u_n''\|_p \leq M$. Since $M^p \geq \int_0^1 |\sigma'(u_n')u_n''|^p \geq \int_0^1 \delta^p |u_n''|^p$, we have

(4.4) $\quad \|u_n''\|_p \leq M/\delta.$

Moreover, since $\|u_n\|_p \leq M'$,

(4.5) $\quad \|u_n'\|_p^p \leq K(\|u_n''\|_p^p + \|u_n\|_p^p) \leq M''$,

where K depends only on p (cf. [1, Lemma 4.1]). Similarly to Lemma 4.3, we obtain from (4.5) that $u \in W^{1,p}(0,1)$. Furthermore by (4.4), we have $u \in W^{2,p}(0,1)$. Again by (4.4) and (4.5),

(4.6) $\quad \|u_n\|_{C^1(0,1)} \leq C\,\|u_n\|_{W^{2,p}} \leq \tilde{M}.$

Especially, since $|u_n'(0)| \leq \tilde{M}$, $u_{n_k}'(0) \to {}^{\exists}w \in R$ for some subsequence $n_k \to \infty$. Then by (4.1) and continuity of σ^{-1}, $u_{n_k}'(x) \to \sigma^{-1}(\sigma(w) - \int_0^x v(\tau)d\tau)$. Noting (4.6), we obtain $u_{n_k}' \to \sigma^{-1}(\sigma(w)-V)$ in $L^p(0,1)$, where $V(x) = \int_0^x v(\tau)d\tau$. Then similarly to Lemma 4.3, we can conclude that A is single-valued and $Au = -\sigma(u')'$ for $u \in D(A)$. The boundary condition is shown in the same way as in Lemma 4.3. Since A is m-accretive, the rest of proof is similar to Lemma 4.4. □

<u>Lemma 4.6.</u> Assume (3.1) and (3.2). Then $(I+A)^{-1}$ is a compact operator in $L^p(0,1)$.

Proof. Suppose first that (3.2) holds for i=0. Let $f \in L^p(0,1)$ and take $u \in D(A)$ such that $u+Au=f$. Then by Lemma 4.4,

(4.7) $\quad \sigma(u'(x)) - \sigma(u'(0)) = \int_0^x u - \int_0^x f .$

Then by the Hölder inequlity and Lemma 4.1, we have $\|u\|_{W^{1,p}} \leq C(\|f\|_p)$. Since the imbedding $W^{1,p}(0,1) \subset L^p(0,1)$ is compact, we conclude that $(I+A)^{-1}$ is compact. If (3.2) holds for i=1, we use $\sigma(u'(1))-\sigma(u'(x))=\int_x^1 u - \int_x^1 f$ instead of (4.7) for $f \in L^p(0,1)$. □

<u>Lemma 4.7.</u> Assume (3.3). Then $(I+A)^{-1}$ is a compact operator in $L^p(0,1)$.

Proof. For $f \in L^p(0,1)$, take $u \in D(A)$ such that $u+Au=f$. Then by (3.3) and Lemma 4.5, $|u''(x)| \leq \frac{1}{\delta}(|u(x)| + |f(x)|)$, so that

$$\|u''\|_p \leq \frac{C}{\delta}(\|u\|_p + \|f\|_p) \leq \frac{2C}{\delta}\|f\|_p \quad \text{(by Lemma 4.1).}$$

Hence $\|u'\|_p^p \leq K(\|u''\|_p^p + \|u\|_p^p) \leq C(\|f\|_p)$ and the rest of proof can be done as in Lemma 4.6. □

<u>Lemma 4.8.</u> Assume either (3.1) and (3.2), or (3.3). Then A satisfies the convergence condition.

Proof. Keeping the Lemmas 4.4 and 4.5 in mind, we observe firstly that $A^{-1}0 \ni u \Leftrightarrow u(x)=$ const and writing $u(x)=u_0$, $0 \in \beta_i(u_0)$ (i=0, 1). ($\Rightarrow$) $\sigma(u')'=0 \Leftrightarrow \sigma(u')=\text{const} \Leftrightarrow u'=\text{const} \Leftrightarrow u(x)=ax+b$. Since $u \in D(A)$, $u'(0) \in \beta_0(u(0))$ and $-u'(1) \in \beta_1(u(1))$, so that $a \in \beta_0(b)$ and $-a \in \beta_1(a+b)$. Since β_i are monotone and $0 \in \beta_i(0)$, we have $a \cdot b \geq 0$ and $-a(a+b) \geq 0$, which implies a=0. Thus u(x)=b and $0 \in \beta_i(b)$. ($\Leftarrow$) trivial.

Above observation shows that if $u \in D(A)$ and $u \notin A^{-1}0$, then $u \neq const$. In fact, if $u \notin A^{-1}0$, then either $u \neq const$ or $u = u_0$ and $0 \notin \beta_i(u_0)$ ($i=0$ or 1). Since $u \in D(A)$, $u = u_0$ implies $0 \in \beta_i(u_0)$. Thus only $u \neq const$ is valid.

Let $P: L^p(0,1) \to A^{-1}0$ be the nearest point mapping, and $u \in D(A)$ and $u \notin A^{-1}0$. Then, by virtue of Lemmas 4.4 and 4.5,

$$\langle Au, J(u-Pu)\rangle \| u-Pu \|^{p-2} = \int_0^1 -\sigma(u')' \operatorname{sgn}(u-Pu)|u-Pu|^{p-1}$$

$$= [-\sigma(u')' \operatorname{sgn}(u-Pu)|u-Pu|^{p-1}]_0^1 + (p-1)\int_0^1 \sigma(u')u'|u-Pu|^{p-2}.$$

The proof of Lemma 4.1 shows that $[\ldots]_0^1 \geq 0$ and the continuity of u' implies the existence of an interval $\Omega \subset (0,1)$ such that $u' \neq 0$ on Ω. Hence $\langle Au, J(u-Pu)\rangle > 0$ is obtained. Combining this with Lemmas 4.6 and 4.7, we conclude that A satisfies the convergence condition by Proposition 2.4. □

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